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Sampling theory approach to prolate spheroidal wavefunctions

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Abstract

We use the Whittaker–Shannon sampling theorem to show that the eigenvalue problem for the sinc-kernel is equivalent to a discrete eigenvalue problem. The well-known eigenfunctions, namely, the prolate spheroidal wavefunctions, their corresponding eigenvalues and the orthogonality and completeness properties are determined without invoking the prolate spheroidal differential equation. This analysis based on the sampling theorem may be used for calculating the eigenvalues and eigenfunctions of bandlimited kernels in general as we illustrate with an additional example of the sinc^2-kernel.

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1. Introduction

Prolate spheroidal wavefunctions of order zero form a complete set of bandlimited functions that are orthogonal over a finite as well as infinite interval. They are eigenfunctions of the finite Fourier transform and also that of the sinc-kernel. The spheroidal wavefunctions are the solutions of the wave equation in spheroidal co-ordinate system [1, 2]. In a series of papers, Slepian *et al* [3–7] extensively investigated the properties of the prolate spheroidal functions and their relation to the uncertainty principle. These functions have found several applications, notably in the theory of laser resonators [8, 9]. Frieden [10] has reviewed the evaluation, design and extrapolation methods for optical signals using prolate functions. These functions also find important applications in the generalized information theory for inverse problems in signal processing [11].

The purpose of this paper is to present a novel approach, based on the Whittaker–Shannon sampling theorem, to the eigenvalue problem for the sinc-kernel. Our analysis also presents an insight into the fact that the number of significant eigenvalues for the sinc-kernel problem is of the order of the space-bandwidth product or the Shannon number of the system. As pointed out by the reviewers, the present work has some similarities with the recent publications of

Walter and Shen [12–14]. In their work, Walter and Shen have used the natural connection between the prolate spheroids and the sinc function to develop several new formulae that are further used to construct filter banks for digitized/sampled versions of bandlimited signals. Our goal in this paper is show that the eigenfunctions for the sinc-kernel, the corresponding eigenvalues and their orthogonality and completeness properties may be determined using the sampling theorem and some related identities, with no reference to the prolate spheroidal differential equation. We also wish to emphasize that the approach presented here is useful for treating the eigenvalue problems associated with general bandlimited kernels. To that effect, an additional illustration of our method for the sinc^2-kernel is provided.

The outline of the paper is as follows. In section 2, we briefly state the Whittaker–Shannon sampling theorem and some related identities, which will be found useful in later sections. The eigenvalue problem for the sinc-kernel is treated with our new sampling theorem based method in section 3. In section 4, we derive the orthogonality and completeness properties of the eigensolutions. Finally in sections 5 and 6, we present some numerical results obtained using the method described in earlier sections. The eigenvalues and eigenfunctions are computed for both the sinc- and the sinc^2-kernels.

2. Whittaker–Shannon sampling theorem

Consider a bandlimited function $g(x)$ in $L^2(-\infty, \infty)$ such that its Fourier transform $G(f)$ is non-zero only in the interval $[-B, B]$ in the frequency domain. Then the sampling theorem [15, 16] allows us to express $g(x)$ in terms of its equally spaced samples as follows:

$$g(x) = \sum_{m=-\infty}^{\infty} g\left(\frac{m}{2B}\right) \operatorname{sinc}(2Bx - m) \tag{1}$$

where

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}. \tag{2}$$

The sinc functions obey the following orthogonality relation:

$$\int_{-\infty}^{\infty} \mathrm{d}x \operatorname{sinc}(2Bx - m) \operatorname{sinc}(2Bx - n) = \frac{1}{2B}\delta_{m,n} \tag{3}$$

where $\delta_{m,n}$ is the Kronecker delta which is zero for $m \neq n$ and equals 1 for $m = n$. As a special case of (1), we take $g(x)$ to be the bandlimited function $\operatorname{sinc}[2B(x - x')]$ and write

$$\sum_{m=-\infty}^{\infty} \operatorname{sinc}(2Bx - m) \operatorname{sinc}(2Bx' - m) = \operatorname{sinc}[2B(x - x')]. \tag{4}$$

The properties (3) and (4) show that the sinc functions centred at $m/(2B)$ for integral values of m form an orthogonal and complete set over $(-\infty, \infty)$ for the class of functions bandlimited to $[-B, B]$. Using (3), we find

$$g\left(\frac{m}{2B}\right) = 2B \int_{-\infty}^{\infty} \mathrm{d}x\, g(x) \operatorname{sinc}(2Bx - m) \tag{5}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}x\, g^2(x) = \frac{1}{2B} \sum_{m=-\infty}^{\infty} g^2\left(\frac{m}{2B}\right). \tag{6}$$

All the above properties may be proved by straightforward calculations and we have stated them here since they will be needed in the following sections.

3. The eigenvalue problem for the sinc-kernel

Consider a one-dimensional imaging system, which passes all the spatial frequencies only in the range $[-B, B]$ with equal weight. When an object $O(x)$ in $[-L, L]$ is imaged using this system, the resultant field distribution $I(x)$ in the image plane is described by

$$I(x) = \int_{-B}^{B} \mathrm{d}f\, \mathrm{e}^{\mathrm{i}2\pi f x} \int_{-L}^{L} \mathrm{d}x'\, \mathrm{e}^{-\mathrm{i}2\pi f x'} O(x')$$
$$= 2B \int_{-L}^{L} \mathrm{d}x'\, \mathrm{sinc}[2B(x - x')] O(x'). \tag{7}$$

The inversion problem of reconstructing $O(x)$ from $I(x)$ has been well studied in the literature in terms of the orthogonal set of eigenfunctions for the sinc-kernel [17]. These eigenfunctions happen to be the prolate spheroidal wavefunctions. The imaging situation described above, however, has no obvious connection with the prolate spheroidal differential equation satisfied by the eigenfunctions. In view of this, it is our aim in this paper to show that it is possible to arrive at the eigenfunctions of the sinc-kernel in an alternative way. The method presented below is direct (non-iterative) and is based on the sampling identities stated in section 2.

We start by considering the following problem:

$$\lambda_n \phi_n(x) = \int_{-L}^{L} \mathrm{d}x'\, \mathrm{sinc}[2B(x - x')] \phi_n(x'). \tag{8}$$

The above equation is a homogeneous Fredholm integral equation of the second kind with $\phi_n(x)$ and λ_n being the eigenfunctions and the associated eigenvalues, respectively. The kernel of the equation is symmetric and square integrable. The properties of eigenfunctions associated with symmetric kernels [18] will not be used for the purpose of the present section where we convert equation (8) to an equivalent discrete eigenvalue problem.

Using (4) we expand $\mathrm{sinc}[2B(x - x')]$ in (8) as a sampling series to get

$$\lambda_n \phi_n(x) = \int_{-L}^{L} \mathrm{d}x' \sum_{m=-\infty}^{\infty} \mathrm{sinc}(2Bx - m)\, \mathrm{sinc}(2Bx' - m) \phi_n(x')$$
$$= \sum_{m=-\infty}^{\infty} \lambda_n \phi_n \left(\frac{m}{2B}\right) \mathrm{sinc}(2Bx - m). \tag{9}$$

The second step above follows from the integral equation (8). From (9), it is clear that the eigenfunctions $\phi_n(x)$ satisfy the sampling theorem (1) and hence are bandlimited to $[-B, B]$. Starting from:

$$\lambda_n \phi_n \left(\frac{m}{2B}\right) = \int_{-L}^{L} \mathrm{d}x'\, \mathrm{sinc}\left[2B\left(\frac{m}{2B} - x'\right)\right] \phi_n(x') \tag{10}$$

we now expand $\phi_n(x')$ as a sampling series to obtain

$$\lambda_n \phi_n \left(\frac{m}{2B}\right) = \int_{-L}^{L} \mathrm{d}x'\, \mathrm{sinc}(2Bx' - m) \sum_{k=-\infty}^{\infty} \phi_n \left(\frac{k}{2B}\right) \mathrm{sinc}(2Bx' - k)$$
$$= \sum_{k=-\infty}^{\infty} A_{mk} \phi_n \left(\frac{k}{2B}\right) \tag{11}$$

where

$$A_{mk} = \int_{-L}^{L} \mathrm{d}x'\, \mathrm{sinc}(2Bx' - m)\, \mathrm{sinc}(2Bx' - k). \tag{12}$$

The integral equation (8) is thus equivalent to the discrete eigenvalue problem for the infinite matrix A. We point out that if we were to replace the sinc-kernel in (8) by a general bandlimited kernel $h(x - x')$ that may be expressed as a sampling series, the conclusion regarding the bandlimited nature of the eigenfunctions will still hold and it is also possible to formulate the discrete eigenvalue problem as in equations (10)–(12). Note that the eigenvalues of equation (8) are the same as that of the matrix A, and the elements of the corresponding discrete eigenvectors are the samples of the continuous eigenfunctions $\phi_n(x)$. The continuous eigenfunctions $\phi_n(x)$ can thus be determined from the discrete eigenvectors of A by using the sampling theorem as in (1). The knowledge of the fact that the solutions of (8) happen to be the prolate spheroidal functions is therefore not necessary for the computation of the eigenvalues and the eigenfunctions. Our study of the literature shows that the computation of prolate spheroids and their eigenvalues is performed using either asymptotic solutions of the prolate spheroidal differential equation [19–22] or Bouwkamp's method of Legendre series [23, 24]. Both these methods rely on the use of the prolate spheroidal differential equation. The sampling theory approach presented above thus provides an interesting alternative to the computation of prolate spheroids. From (12) we see that the matrix elements A_{mk} fall off to zero as the main lobes of the corresponding sinc-functions go beyond the range of integration $[-L, L]$, i.e. when $|m|, |k| > 2BL$. Clearly, only a square sub-matrix of A with the dimension of the order of the space-bandwidth product $(4BL)$ has elements with significant magnitude, so that, when calculating the eigenvalues by equating determinant$(A - \lambda I)$ to zero, it is clear that the number of significant eigenvalues for the sinc-kernel is at most of the order of $(4BL)$, the remaining eigenvalues being close to zero. This property has important implications in the solution of inverse problems (see, for example, [11]). Later in section 5 we will compare the trace of the matrix A with the sum of highest $c \sim (4BL)$ (Shannon number) calculated eigenvalues to illustrate this point. A much more quantitative analysis of the number of significant eigenvalues is presented in [5, 14].

4. Orthogonality and completeness properties of the eigenfunctions

The orthogonality and completeness properties of the prolate spheroids are well known and have been treated in detail before [3]. However, in the spirit of this paper we take a different route to proving them, once again by means of the sampling identities stated in section 2. We note that the proofs of orthogonality and completeness properties given below are specific to the sinc-kernel problem. For brevity, we denote the discrete eigenvectors of the matrix A by $u_n = \left[\ldots \phi_n\left(\frac{m}{2B}\right) \ldots\right]^T$, with T standing for the transpose of the row vector. The matrix A in (12) is real symmetric, so that its eigenvalues and eigenvectors must be real. Clearly, the eigenfunctions $\phi_n(x)$ obtained from the elements of u_n using the sampling theorem are also real. Note that the matrix A is also centrosymmetric with respect to the element A_{00}. It is known that the eigenvectors of symmetric centrosymmetric matrices are either symmetric or skew symmetric [25]. The corresponding continuous eigenfunctions $\phi_n(x)$ are therefore either even or odd. We assume the eigenfunctions $\phi_n(x)$ to be normalized to 1 over $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \phi_n^2(x) = 1 = \frac{1}{2B} \sum_{m=-\infty}^{\infty} \phi_n^2\left(\frac{m}{2B}\right). \tag{13}$$

The second equality above follows from (6) since $\phi_n(x)$ is bandlimited. The real symmetric nature of A assures that the matrix A may be factored in form $A = U A_d U^T$ such that A_d is a diagonal matrix with U obeying:

$$U^T U = 2B\, \hat{1} = U U^T. \tag{14}$$

The columns of U are formed by the eigenvectors u_n. The factor $2B$ above occurs due to the choice of normalization in (13). The non-degeneracy of eigenvalues for the sinc-kernel is known [3] and this fact along with the nature of A once again confirms (14). Equation (14) may be written explicitly in terms of the samples of eigenfunctions as follows:

$$\sum_{m=-\infty}^{\infty} \phi_n\left(\frac{m}{2B}\right)\phi_l\left(\frac{m}{2B}\right) = 2B\delta_{n,l} \tag{15}$$

and

$$\sum_{n=0}^{\infty} \phi_n\left(\frac{m}{2B}\right)\phi_n\left(\frac{k}{2B}\right) = 2B\delta_{m,k}. \tag{16}$$

Equations (15) and (16) above are simply the restatements of the first and the second equalities in (14), respectively. Walter and Shen have obtained equations (15) and (16) in their recent publications [12–14] using the well-known properties of prolate spheroids. Here, we have arrived at these relations independently as a consequence of the real symmetric nature of the matrix$A$ defined in (12). We now proceed to prove the orthogonality and completeness properties of the eigensolutions.

4.1. Orthogonality and completeness over the bandlimited subspace of $L^2(-\infty,\infty)$

Using the sampling theorem (1) and the orthogonality condition (3), we can write

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}x\, \phi_n(x)\phi_l(x) &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \phi_n\left(\frac{m}{2B}\right)\phi_l\left(\frac{k}{2B}\right)\frac{1}{2B}\delta_{m,k} \\ &= \frac{1}{2B}\sum_{m=-\infty}^{\infty} \phi_n\left(\frac{m}{2B}\right)\phi_l\left(\frac{m}{2B}\right) \\ &= \delta_{n,l}.\end{aligned} \tag{17}$$

The last step in (17) follows from (15). The orthogonality of the discrete eigenvectors is thus equivalent to the orthogonality of the continuous eigenfunctions over $(-\infty,\infty)$. Now consider

$$\sum_{n=0}^{\infty}\phi_n(x)\phi_n(x') = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left[\sum_{n=0}^{\infty}\phi_n\left(\frac{m}{2B}\right)\phi_n\left(\frac{k}{2B}\right)\right]\mathrm{sinc}(2Bx-m)\,\mathrm{sinc}(2Bx'-k). \tag{18}$$

The sum over index n is simplified by (16) and additionally using (4) we get

$$\sum_{n=0}^{\infty}\phi_n(x)\phi_n(x') = 2B\,\mathrm{sinc}[2B(x-x')]. \tag{19}$$

The above equation is just a restatement of the Mercer's theorem [18] with normalization of eigenfunctions as defined in (13). For an arbitrary function $g(x)$ that is bandlimited to $[-B, B]$, we therefore have

$$\int_{-\infty}^{\infty}\mathrm{d}x\, g(x)\sum_{n=0}^{\infty}\phi_n(x)\phi_n(x') = g(x'). \tag{20}$$

Thus, completeness of the eigensolutions over the class of functions in $L^2(-\infty,\infty)$ bandlimited to $[-B, B]$ is proved since an arbitrary bandlimited function $g(x)$ of this class may be represented as a linear combination of eigenfunctions $\phi_n(x)$.

4.2. Orthogonality and completeness over $L^2(-L, L)$

The dual orthogonality over infinite as well as finite intervals is a special property of the eigensolutions. It is well known that the eigenfunctions of a symmetric kernel associated with distinct eigenvalues are orthogonal over the range of integration $(-L, L)$. Nevertheless, we prove this property below using sampling identities. Consider:

$$\begin{aligned}\int_{-L}^{L} \mathrm{d}x\, \phi_n(x)\phi_l(x) &= \sum_{m=-\infty}^{\infty} \phi_n\left(\frac{m}{2B}\right) \int_{-L}^{L} \mathrm{d}x\, \mathrm{sinc}(2Bx - m)\phi_l(x) \\ &= \lambda_l \sum_{m=-\infty}^{\infty} \phi_n\left(\frac{m}{2B}\right) \phi_l\left(\frac{m}{2B}\right) \\ &= 2B\lambda_l \delta_{n,l}. \end{aligned} \tag{21}$$

To prove completeness over $L^2(-L, L)$ we start by writing equation (8) as

$$2B\lambda_n\phi_n(x) = \int_{-L}^{L} \mathrm{d}x'\, \phi_n(x') \int_{-B}^{B} \mathrm{d}f'\, \mathrm{e}^{\mathrm{i}2\pi f'(x-x')}. \tag{22}$$

Denoting the finite Fourier transform of $\phi_n(x)$ by

$$\Phi_n(f) = \int_{-L}^{L} \mathrm{d}x\, \phi_n(x)\, \mathrm{e}^{-\mathrm{i}2\pi f x} \tag{23}$$

and using (22), we see that the finite Fourier transform satisfies:

$$\frac{B}{L}\lambda_n \Phi_n(f) = \int_{-B}^{B} \mathrm{d}f'\, \Phi_n(f')\, \mathrm{sinc}[2L(f - f')]. \tag{24}$$

Equation (24) above is simply a scaled version of equation (8) and we thus conclude that

$$\Phi_n(f) = \int_{-L}^{L} \mathrm{d}x\, \phi_n(x)\, \mathrm{e}^{-\mathrm{i}2\pi f x} = \alpha_n \phi_n\left(\frac{Lf}{B}\right). \tag{25}$$

Here, α_n is a constant for the particular $\phi_n(x)$under consideration. The factor (L/B) in the argument of the last term above adjusts the scaling appropriately. It may be shown that $\alpha_n = i^n(2L\lambda_n)^{1/2}$, however, this value is of no consequence in what follows.

Using (25) and the identity (19), we observe that

$$\int_{-L}^{L} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}2\pi f x} \sum_{n=0}^{\infty} \frac{1}{\alpha_n}\phi_n(x)\phi_n\left(\frac{Lf'}{B}\right) = 2B\, \mathrm{sinc}[2L(f - f')]. \tag{26}$$

This result suggests the expansion:

$$\frac{L}{B}\sum_{n=0}^{\infty} \frac{1}{\alpha_n}\phi_n(x)\phi_n\left(\frac{Lf'}{B}\right) = \exp(\mathrm{i}2\pi f'x) \quad |x| < L. \tag{27}$$

This result has also been obtained in [24]. We now note that any function $p(x)$ in $L^2(-L, L)$ may be expressed as a Fourier expansion:

$$p(x) = \int_{-\infty}^{\infty} \mathrm{d}f\, P(f)\, \mathrm{e}^{\mathrm{i}2\pi f x} \tag{28}$$

and in view of (27) can thus be expressed as a linear combination of $\phi_n(x)$. The completeness in $L^2(-L, L)$ is thus proved. It is interesting to note that, the occurrence of the sinc-function as the kernel of the integral equation as well as the interpolation function in the sampling formula plays an important role in the orthogonality and completeness proofs above. This offers an explanation to the fact that among the functions bandlimited to $[-B, B]$, only the

Table 1. Ten largest eigenvalues for the sinc-kernel calculated using the sampling theorem based method for $c = 10$. The calculations are performed with the dimension N of the truncated matrix A taken equal to 51, 201 and 501, respectively. For maintaining consistency with the literature [19], values of $(2B\lambda_n)$ are given in the table below.

n	$N = 51$	$N = 201$	$N = 501$
0	1.0000×10^{000}	1.0000×10^{000}	1.0000×10^{000}
1	1.0000×10^{000}	1.0000×10^{000}	1.0000×10^{000}[a]
2	9.9989×10^{-001}	9.9989×10^{-001}	9.9989×10^{-001}
3	9.9767×10^{-001}	9.9784×10^{-001}	9.9788×10^{-001}
4	9.7444×10^{-001}	9.7446×10^{-001}	9.7446×10^{-001}
5	8.1176×10^{-001}	8.2173×10^{-001}	8.2377×10^{-001}
6	4.4015×10^{-001}	4.4015×10^{-001}	4.4015×10^{-001}
7	1.0194×10^{-001}	1.0973×10^{-001}	1.1129×10^{-001}
8	1.4903×10^{-002}	1.4920×10^{-002}	1.4920×10^{-002}
9	1.1073×10^{-003}	1.2655×10^{-003}	1.2952×10^{-003}

[a] The eigenvalues corresponding to $n = 0, 1$ are not degenerate. Our calculations show that they differ in sixth significant digit. All the eigenvalues are less than one.

prolate spheroids enjoy the curious dual orthogonality property over finite as well as infinite intervals [3, 10]. We conclude this section by calculating the trace of the matrix A, which equals the total sum of its eigenvalues.

$$\mathrm{tr}(A) = \int_{-L}^{L} \mathrm{d}x \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(2Bx - m) = 2L. \tag{29}$$

We have used the special case $x = x'$ of the identity (4) for evaluation of the summation above.

5. Numerical computation of eigenfunctions and eigenvalues for the sinc-kernel

In this section, we illustrate the sampling theorem based method presented in section 3 for computation of the eigenfunctions. The Shannon number c in the literature on prolate spheroids [3] in our notation equals $2\pi BL$. We choose $L = 1$ and determine the eigenvalues and eigenfunctions for the special case $c = 10$. The reason for choosing this particular value of c is that, in the literature [1, 10, 21, 22], different asymptotic expansions are suggested for calculating the spheroidal functions depending on whether $c < 10$ or $c > 10$. Our method explained before is independent of the magnitude of c, apart from requiring the dimension of the truncated version of matrix A to be sufficiently larger than c. The matrix elements A_{mk} defined in (12) were first calculated by numerical integration using Gaussian quadrature (with the tolerance set to 10^{-10}) and the eigenvalue problem $Au_n = \lambda_n u_n$ was solved for a truncated version of the matrix A. The eigenvectors thus obtained were then interpolated with sinc-function according to the sampling theorem (1). We have calculated the first ten eigenvalues for truncated versions of matrix A with dimension $N = 51, 101, 151, 201, \ldots, 501$. In table 1, we show the ten largest eigenvalues for the cases $N = 51, 201, 501$, respectively. We observe that for $N = 151$ and beyond, the eigenvalues corresponding to the even orders have stabilized up to five significant digits and match with those obtained by Slepian and Sonneblick [19] using asymptotic expansions. The eigenvalues corresponding to the odd orders barring $n = 1$, on the other hand, have kept increasing progressively in third or fourth significant digit. For $N = 501$, the eigenvalues corresponding to orders $n = 3, 5, 7, 9$ match those in [19] up to

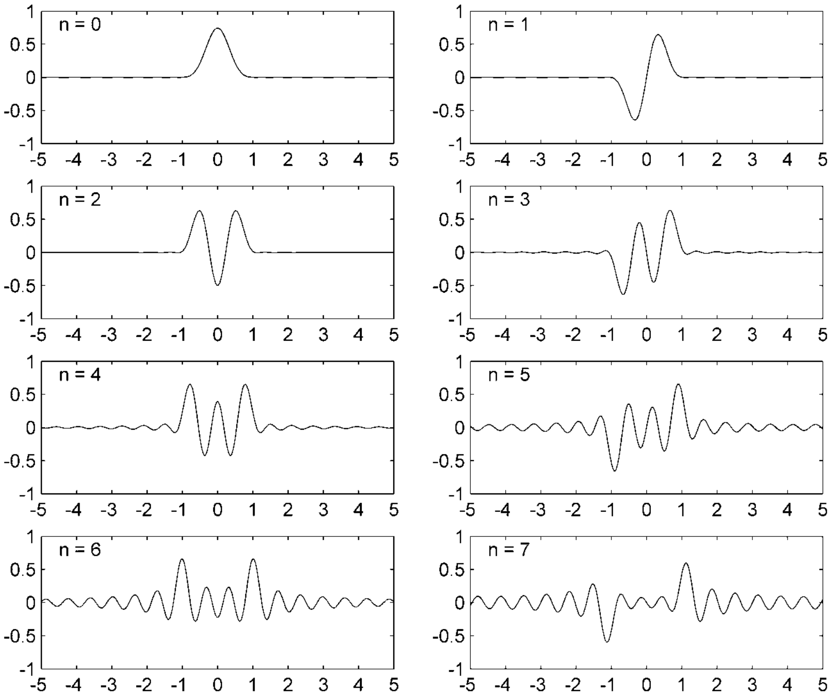

Figure 1. First eight eigenfunctions for the sinc-kernel calculated using the sampling theorem based method. The functions are even/odd for even/odd index n. The eigenfunction corresponding to index n has n zeros in the interval $[-1, 1]$.

4, 2, 2 and 2 significant digits, respectively. A further analysis is necessary for deciding the dimension of the matrix A for given accuracy requirements and the topic is beyond the scope of this paper. The sharp fall off in the eigenvalues beyond $n > 2c/\pi$ is evident. The sum of the first ten eigenvalues for all the cases $N = 51, \ldots, 501$, respectively is over 99% of the value of the trace as determined by equation (29). The first eight eigenfunctions calculated by interpolating the corresponding discrete eigenfunctions for the case $N = 501$ are shown in figure 1. As is well known, one sees that the functions $\phi_n(x)$ are even/odd for even/odd values of n. Also, $\phi_n(x)$ has n zeros in $[-1, 1]$.

6. Eigenvalue problem for the sinc^2-kernel

In this section we illustrate the general nature of our sampling theorem based method with an example of the eigenvalue problem for the sinc^2-kernel:

$$\mu_n \psi_n(x) = \int_{-L}^{L} \mathrm{d}x' \, \mathrm{sinc}^2[B(x - x')] \psi_n(x). \tag{30}$$

This problem has been studied before in the context of incoherent imaging systems [26 and references therein] using iterative methods to obtain an orthogonal set of eigenfunctions over $[-L, L]$ and approximate expressions for the eigenvalues have been presented. We believe that a direct method based on sampling theorem for general bandlimited kernels has not appeared in the literature. We have used B instead of $2B$ in the argument of sinc^2 in order to maintain the same bandwidth as in the sinc example. Following the analysis parallel to that presented

Table 2. Ten largest eigenvalues for the sinc^2-kernel calculated using the sampling theorem based method for $c = 10$. The calculations are performed with the dimension N of the truncated matrix A taken equal to 51, 201 and 501, respectively. Values of $(B\mu_n)$ are given in the table below.

n	$N = 51$	$N = 201$	$N = 501$
0	8.8802×10^{-001}	8.8802×10^{-001}	8.8802×10^{-001}
1	7.3441×10^{-001}	7.3441×10^{-001}	7.3441×10^{-001}
2	5.8579×10^{-001}	5.8579×10^{-001}	5.8579×10^{-001}
3	4.3822×10^{-001}	4.3822×10^{-001}	4.3822×10^{-001}
4	2.9652×10^{-001}	2.9652×10^{-001}	2.9652×10^{-001}
5	1.6363×10^{-001}	1.6363×10^{-001}	1.6363×10^{-001}
6	6.1095×10^{-002}	6.1097×10^{-002}	6.1097×10^{-002}
7	1.3420×10^{-002}	1.3426×10^{-002}	1.3426×10^{-002}
8	1.8151×10^{-003}	1.8132×10^{-003}	1.8132×10^{-003}
9	1.6746×10^{-003}	1.6846×10^{-003}	1.6848×10^{-003}

in equations (10)–(12), it is easy to see that for the solution of (30), one needs to solve the equivalent discrete eigenvalue problem for the infinite matrix A' with elements defined by

$$A'_{mk} = \int_{-L}^{L} \mathrm{d}x' \, \mathrm{sinc}^2(Bx' - m/2)\, \mathrm{sinc}(2Bx' - k). \tag{31}$$

The properties of the eigenfunctions $\psi_n(x)$will not be discussed here, as they have already appeared in literature [26] and also since the purpose of this example is simply to illustrate the application of the sampling theorem based method to kernels other than the sinc-kernel. We however note that, the eigenfunctions $\psi_n(x)$ do not enjoy the special properties of dual orthogonality, as is the case with prolate spheroids. To calculate the trace of matrix A', we write the sampling expansion for the sinc^2 function:

$$\mathrm{sinc}^2[B(x - x')] = \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(Bx' - m/2)\, \mathrm{sinc}(2Bx - m). \tag{32}$$

Using the special case $x = x'$ of (32), we can write

$$\mathrm{tr}(A') = \int_{-L}^{L} \mathrm{d}x \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(Bx - m/2)\, \mathrm{sinc}(2Bx - m) = 2L. \tag{33}$$

For numerical computations we have used the same values $L = 1$, $c = 2\pi BL = 10$ as in the previous section. The matrix elements as defined in (31) were once again computed using a Gaussian quadrature and the eigenvalue problem $A'w_n = \mu_n w_n$ was solved for the truncated versions of matrix A' having dimension $N = 51, 101, 151, \ldots, 501$. The ten largest eigenvalues for each of the three cases $N = 51, 201, 501$ are shown in table 2. We observe that for the cases $N = 301$ and beyond, all the ten eigenvalues have stabilized to five significant digits. The sum of the first ten eigenvalues for all the cases $N = 51, \ldots, 501$ is over 99% of the value of trace as determined by equation (33). The discrete eigenvectors w_n corresponding to the case $N = 501$ are interpolated using the sampling theorem (1) and the first eight continuous eigenfunctions are plotted in figure 2. The eigenfunctions are qualitatively similar to the sinc-kernel problem and the eigenvalues are seen to fall off almost linearly with the order index n.

All the computations in sections 5 and 6 were performed using standard routines available in the mathematical software MATLAB v. R12.

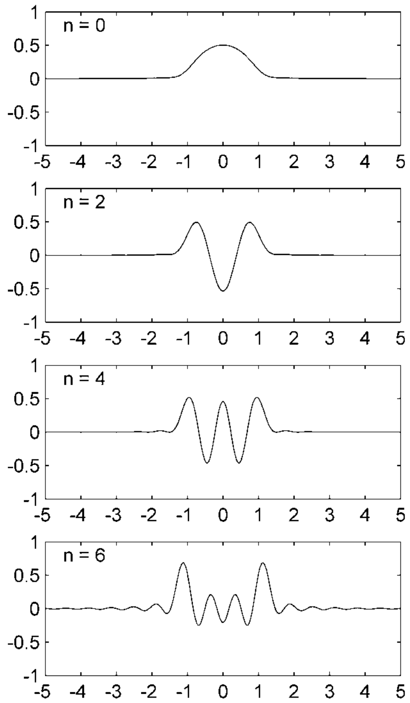

Figure 2. First eight eigenfunctions for the sinc^2-kernel calculated using the sampling theorem based method. The functions are even/odd for even/odd index n. The eigenfunction corresponding to index n has n zeros in the interval $[-1, 1]$.

7. Conclusion

In summary, we have treated the eigenvalue problem for the sinc-kernel with a novel approach, based on the Whittaker–Shannon sampling theorem. First, the eigenvalue problem (8) is converted to an equivalent discrete eigenvalue problem for the infinite matrix A defined in (12). It is shown that the eigenfunctions (prolate spheroidal wavefunctions) and the associated eigenvalues may be computed without any reference to the prolate spheroidal differential equation. The orthogonality and completeness properties of the eigenfunctions are also proved using sampling identities. It is pointed out that the appearance of the sinc-function as a kernel of the integral equation as well as the interpolation function in the sampling formula is important for the dual orthogonality of prolate spheroidal functions over finite as well as infinite intervals. The sampling theorem based method is illustrated with the computation of eigenfunctions and eigenvalues for the sinc-kernel with Shannon number $c = 10$. The method of calculating eigenfunctions and eigenvalues presented in this paper is applicable to general bandlimited kernels as illustrated by the sinc^2-kernel example.

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